# Digital Intersections: minimal carrier, connectivity and periodicity properties

Isabelle Sivignon [a,*], Florent Dupont [b], Jean-Marc Chassery [a]

[a] *Laboratoire LIS - Grenoble*
*UMR 5083 CNRS*
*961, rue de la Houille Blanche*
*38402 St Martin D'Hères, France*

[b] *Laboratoire LIRIS - Université Claude Bernard Lyon 1*
*FRE 2672 CNRS*
*Bâtiment Nautibus - 8 boulevard Niels Bohr*
*69622 Villeurbanne Cedex, France*

* Corresponding Author. E-mail: sivignon@lis.inpg.fr - Tel: +33 476826467 - Fax: +33 476826384

**Abstract**

Digital geometry is very different from Euclidean geometry in many ways and the intersection of two digital lines or planes is often used to illustrate those differences. Nevertheless, while digital lines and planes are widely studied in many areas, very few works deal with the intersection of such objects. In this paper, we investigate the geometrical and arithmetical properties of those objects. More precisely, we give some new results about the connectivity, periodicity and minimal parameters of the intersection of two digital lines or planes.

*Key words:* digital geometry, digital line, digital plane, intersection

## 1 Introduction

Digital straight lines and digital planes properties have been widely studied in many fields like topology, geometry and arithmetics. Topologically, those objects are well defined according to the digitization scheme used. On the geometrical ground, connectivity features are known and a characterization using convex hull properties [1] has been proposed. Finally, an arithmetical definition [2,3] provides a general model to handle all the definitions proposed so far.

Those properties led to many recognition algorithms. Geometric algorithms [4] decide whether a set of pixels/voxels is a digital line/plane or not, and arithmetical algorithms [5] also return, for a given digitization scheme, the parameters of the Euclidean lines/planes containing the set of pixels/voxels in their digitization.

Discrete geometry is different from Euclidean geometry in many ways, and the differences between the intersection of two Euclidean lines and two digital lines is often used to illustrate this difference. Indeed, while the intersection of two Euclidean lines is a Euclidean point, the intersection of two digital lines can be a discrete point, a set of discrete points or even empty on rectangular grids. Examples of digital lines intersection are depicted in Figure 1.

In [6], an enumeration algorithm of the intersection pixels is proposed using quasi-affine applications. In [7], using the arithmetical definition of a discrete line/plane, Debled et al. present a definition of the set of intersection pixels/voxels of two digital lines/planes using an unimodular matrix. This definition enables the design of an efficient algorithm to determine all the pixels/voxels of an intersection given the parameters of the two lines/planes. However, no results are given about the topology and arithmetics of this intersection.

Our main contribution is summarized in two points:

- Analysis of the connectivity of two digital lines intersection;
- Characterization and algorithmic identification of the minimal digital line/plane containing the set of pixels/voxels belonging to two digital lines/planes.

The paper is organized as follows: in section 2, we recall some definitions and properties about the rational fractions which are the mathematical framework used in this paper. Section 3 deals with the intersection of two digital lines. We present a criterion to analyze the connectivity of the intersection of any two digital lines, thus completing the results presented in [2] for lines with slopes between 0 and 1. Then, we propose a study about the minimal arithmetic parameters of digital lines intersection and we design an efficient algorithm to find those parameters. To conclude this part on digital lines intersection, a discussion about the extent of the intersection is proposed. In section 4 we present some results on digital planes intersection: we prove that the intersection is periodic and give the minimal period. Finally, we define and determine the minimal parameters of the intersection of two digital planes.

## 2 Rational Fractions

We recall in this part some mathematical definitions and properties about rational fractions needed for this work. The link between those arithmetical structures and digital lines will be exposed further in the paper.

### *2.1 Decomposition into Continued Fractions*

The decomposition of a rational fraction into continued fractions is a mean to describe a floating number using integers. Let us consider a rational fraction $\frac{a}{b}$ where $a$ and $b$ are two relatively prime integers. Then we call *decomposition into continued fractions* of $\frac{a}{b}$ and denote $[q_0, q_1, q_2, \ldots, q_n]$ the integers such that $\frac{a}{b} = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \ldots + \frac{1}{q_{n-1} + \frac{1}{q_n}}}}$. The quotients $q_i$ are the integers given by Euclid's algorithm applied with $a$ and $b$. For instance, the decomposition into continued fractions of $\frac{3}{5}$ is $[0, 1, 1, 2]$. Similarly, we have $\frac{5}{3} = [1, 1, 2]$.

The decomposition into continued fractions can also be computed for an irrational number, and in this case, the sequence of integers is infinite. In this paper, we are only interested in rational fractions since real numbers are strangers to finite discrete geometry.

### *2.2 Farey Series and Stern-Brocot Tree*

The Farey series and the Stern-Brocot tree are two different methods to enumerate and represent all the positive rational fractions $\frac{a}{b}$ where $a$ and $b$ are relatively prime. The definitions proposed in this section can be found for instance in [8] or [9].

The *Farey series* of order $N$, denoted $\mathcal{F}_N$, is the series of ordered irreducible rational fractions between 0 and 1 the with a denominator lower or equal to $N$. For instance, we have: $\mathcal{F}_5 = \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$.

This series is computed from $\mathcal{F}_1 = \frac{0}{1}, \frac{1}{1}$ as follows: the *median* of two rational fractions $\frac{u}{v}$ and $\frac{u'}{v'}$ is defined as the rational fraction $\frac{u+u'}{v+v'}$; the Farey series of order $q$ is computed iteratively from the Farey series of order $q-1$ adding all the medians with a denominator lower or equal to $q$ of consecutive fractions of $\mathcal{F}_{q-1}$. Remark that if $\frac{u}{v}$ and $\frac{u'}{v'}$ are two consecutive fractions of $\mathcal{F}_N$ then we have $u'v - uv' = 1$ (This property will be useful in Section 3.2.2).

The *Stern-Brocot tree* is another way to represent all the irreducible rational fractions (see [9] for a complete definition or [10] for a more informal approach). An illustration of this tree is proposed in Figure 6(a). The idea behind its construction is to begin with the two fractions $\frac{0}{1}$ and $\frac{1}{0}$ and to repeat the insertion of the median of these two fractions. We call *mothers* of a fraction $\frac{u}{v} \in \mathcal{F}_q$ the two fractions $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ of $\mathcal{F}_{q-1}$ such that $\frac{u_1+u_2}{v_1+v_2} = \frac{u}{v}$.

Any positive irreducible rational fraction appears exactly one time in the Stern-Brocot tree. This enables to map a binary code to any irreducible rational fraction which corresponds to the path from the fraction $\frac{1}{1}$ to the rational fraction. Indeed, the unique path leading to a rational fraction $\frac{a}{b}$ can be defined with a sequence of left (code 0) and right (code 1) directions in the tree. This encoding is a one-to-one and onto transformation from a rational fraction to a binary word. For instance, the code 0011 means that we choose two times the left son and then two times the right son in the tree, which leads the the fraction $\frac{3}{7}$. This representation raises two problems:

- how to compute the fraction corresponding to a given code ?
- how to compute the code corresponding to a given fraction ?

In [8], Graham et al. propose the two algorithms detailed in Algorithms 1 and 2 to solve those two problems. Algorithm 1 uses the fact that the Stern-Brocot tree is a binary search tree (BST for short). Indeed, the algorithm is very similar to the classical BST Search algorithm [11]. Algorithm 2 computes recursively the two mothers of the rational fraction associated to a given code. Those two fractions are represented with a $2 \times 2$ matrix which is initialized to the identity matrix (representation of the two fractions $\frac{0}{1}$ and $\frac{1}{0}$) when the function is called. Nevertheless, those algorithms use arithmetical properties of the construction of the tree that we will not detail here. The interested reader may read [8] for further explanations.

Finally, let us point out the link between the Stern-Brocot tree, the binary encoding of rational fractions and the decomposition into continued fractions defined in the previous section. Indeed if we denote $f$ the function which maps a binary word to a rational fraction, we have:

$$f(1^{q_0}0^{q_1}1^{q_2}\dots 1^{q_{n-1}}) = q_0 + \cfrac{1}{q_1 + \frac{1}{q_2+\ldots+\frac{1}{q_{n-1}+1}}} \qquad (1)$$

**Algorithm 1** Compute the binary code associated to a rational fraction $\frac{a}{b}$

```
COMPUTE_CODE(a,b)
C = ∅;
while ( a ≠ b) do
   if (a < b) then
      append(C,0); b = b − a;
   else
      append(C,1); a = a − b;
   end if
end while
return C;
```

**Algorithm 2** Compute the rational fraction associated to a binary code C

```
COMPUTE_FRACTION(C, k, i, res)
L = (1 1; 0 1); R = (1 0; 1 1);
if (i=k) then
   return res; {res is initially set to the identity 2 × 2 matrix}
else if (C[i] = 0) then
   Compute_Fraction(C,k,i+1,res*L);
else
   Compute_Fraction(C,k,i+1,res*R);
end if
```

where $L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

In this equation, the notation $\alpha^q$ denotes $q$ successive repetitions of the letter $\alpha$ in the binary word. For instance, $f(0011) = f(1^0 0^2 1^2) = 0 + \frac{1}{2+\frac{1}{2+1}} = \frac{3}{7}$. Another algorithm to compute the fraction associated to a binary code may be derived from this remark.

## 3 Digital Lines Intersection

In this section, we focus on the properties of digital lines intersections. A digital naive line of parameters $(a, b, \mu)$ is the set of integer points $\{(x, y)\}$ fulfilling the conditions $0 \leq ax - by + \mu < \omega$. $\omega$ is called the thickness of the digital line. In this work, we focus on the thinnest 8-connected lines, called naive lines, with $\omega = \max(|a|, |b|)$. An illustration is proposed in Figure 1(a). Such lines can be also defined using the Chain code depicted on Figure 1(b). This encoding defines a set of eight directions that are used to describe the movements between successive pixels of the digital line. A classical result is that the Chain code of any digital naive line is composed of at most two consecutive different directions. Thus, one can define 8 octants from those eight directions, one digital line belonging to one octant. Nevertheless, only 4 octants remain if we consider symmetries around the central point. For instance, the octant $\{4, 5\}$ is equivalent to the octant $\{0, 1\}$.

### *3.1 Connectivity*

Let us consider two digital naive lines denoted $L_1$ and $L_2$. $L_1 \cap L_2$ is a set of pixels whose connectivity depends on the parameters of the two digital lines.

In 1991, J.-P. Reveillès [2] proposed a criterion to determine whether the intersection of two digital naive lines with slopes between 0 and 1 is connected or not. However, he did not give any information about the intersection of any two digital naive lines. First of all, let us recall the result of Reveillès in the following theorem.

**Theorem 1 ([2])** *Let $L_1$ and $L_2$ be two digital naive lines with rational slopes $\frac{a}{b}$ and $\frac{a'}{b'}$, $0 < \frac{a'}{b'} < \frac{a}{b} \leq 1$. Suppose that the development into continued fractions of $\frac{a}{b}$ is given by $\frac{a}{b} = [q_0, q_1, q_2, \ldots, q_n]$. Then, $L_1 \cap L_2$ is 8-connected if and only if one of the following inequalities holds:*

$$\frac{a'}{b'} < \begin{cases} \frac{1}{2q_1 - 1} \text{ if } q_1 \geq 1 \text{ and } q_2 = 0 \\ \frac{1}{2q_1 + 1} \text{ if } q_1 \geq 1 \text{ and } q_2 = 1 \\ \frac{1}{2q_1} \text{ if } q_1 \geq 2 \text{ and } q_2 \geq 2 \\ \frac{q_2}{q_2 + 2} \text{ if } q_1 = 1 \text{ and } q_2 \geq 2 \end{cases}$$

Note that the coefficient $q_0$ is zero since $0 \leq \frac{a}{b} < 1$. This condition defines a neighborhood $\mathcal{N}$ of the rational fraction $\frac{a}{b}$ such that the intersection is simply 8-connected if and only if $\frac{a'}{b'}$ does not belong to $\mathcal{N}$. Indeed, if the slopes of the two lines are too close, the intersection may be not simply connected. With the following proposition, we describe the connectivity of the intersection of two digital lines lying in two different octants.

**Proposition 2** *Let $L_1$ and $L_2$ be two digital naive lines. Then:*

- *if they belong to the same octant, their intersection may be not connected, and Theorem 1[2] gives a criterion to analyze exactly the connectivity;*
- *if they belong to two neighbor octants, their intersection is either empty or connected;*
- *otherwise, their intersection is either empty or reduced to a unique pixel.*

In the following we denote $F_1$ (resp. $F_2$) the set of directions composing the Chain code of $L_1$ (resp. $L_2$). An illustration is given Figure 1.

**PROOF.** Let $L_1$ and $L_2$ be two digital naive lines. If $L_1$ and $L_2$ belong to the same octant, $|F_1 \cap F_2| = 2$. If they belong to neighbor octants, $|F_1 \cap F_2| = 1$. Otherwise $|F_1 \cap F_2| = 0$. Without loss of generality, we suppose that $L_1$ belongs to the octant $\{0, 1\}$. Let us give a classification of the pixels of $L_1$ and $L_2$. We denote $p_{1,k} = p_{2,k} = p_k$ the pixel of $L_1 \cap L_2$ with minimal x-coordinate if there exist one (there is at most one pixel verifying the property since a digital line of the octant $\{0, 1\}$ does not contain two pixels with the same x-coordinate). Then, $p_{l,k+1}$ is the successor of $p_k$ along $L_l$ according to the Chain code $F_l$.

- if $|F_1 \cap F_2| = 0$ (Figure 1(c)), then $p_{1,k+1} \neq p_{2,k+1}$ as they are the successors of the same point using two different directions. Suppose that $L_2$ is composed of 2 and 3. The other cases are symmetrical. Then, let us consider a pixel $p_1(x_p, y_{p_1}) \in L_1$ with $x_p$ greater that the $x$-coordinate $x_k$ of $p_k$, and $p_2(x_p, y_{p_2}) \in L_2$. Then, $y_{p_1} \geq y_k$ and $y_{p_2} \leq y_k - (x_p - x_k) \leq y_k$, since $x_p \geq x_k$. Hence, the two lines do not have any common point after $p_k$.
- if $|F_1 \cap F_2| = 1$ (Figure 1(d)) then let us denote $\alpha_{1i}$ (resp. $\alpha_{2i}$) the direction used from $p_{1,i}$ to $p_{1,i+1}$ (resp. $p_{2,i}$ to $p_{2,i+1}$). Hence, while $\alpha_{1i} = \alpha_{2i}$, $i \geq k$, $p_{1,i+1} = p_{2,i+1}$. Both pixels $p_{1,i}$ and $p_{1,i+1}$ belong to the intersection and are 8-connected. Unless the two lines are confounded, there exist $j$ such that $\alpha_{1j} \neq \alpha_{2j}$. Hence, $p_{1,j+1} \neq p_{2,j+1}$. Suppose that $L_2$ is composed of 1 and 2. The other cases are symmetrical. Then, let us consider a pixel $p_1(x_p, y_{p_1}) \in L_1$ with $x_p$ greater that the $x$-coordinate $x_j$ of $p_{1,j} = p_{2,j}$, and $p_2(x_p, y_{p_2}) \in L_2$. Then, $y_{p_1} \leq y_j + (x_p - x_j - 1)$ and $y_{p_2} \geq y_j + (x_p - x_j)$. Hence, the two lines do not have any common point after $p_{1,j}$.
- if $|F_1 \cap F_2| = 2$ (Figure 1(e)) , we refer to [2] to analyze the connectivity.

□

Figure 2 summarizes the connectivity of the intersection between a given digital line with slope $0 \leq \frac{a}{b} \leq 1$ and any other digital line with a rational slope.

### *3.2 Minimal Parameters*

The intersection of two digital lines is a set of collinear discrete points. To characterize this set of points, it is interesting to know the straight lines containing all the intersection pixels in their digitization. Obviously, the two lines we are studying are solutions.

**Definition 3 (minimal parameters)** *Let $\mathcal{P}$ be a given set of discrete points and let $S$ be the set of parameters of naive straight lines containing $\mathcal{P}$. We have $S = \{(a, b, \mu) \mid \mathcal{P} \subset \{(x, y) \mid 0 \leq ax - by + \mu < max(|a|, |b|)\}\}$. Then the* *minimal parameters of $\mathcal{P}$ are the values $(a, b, \mu)$ of $S$ with minimal $max(|a|, |b|)$ and minimum $\mu$.*

In other words, the minimal parameters are chosen among the parameters of the digital lines which contain the intersection pixels. An illustration of this definition is given in Figure 3: the digital line represented with black squares contains all the pixels (double squared) of the two other lines intersection.

In the following, we propose a characterization and an algorithm to find the minimal parameters of the intersection of any two digital naive lines using two different methods and emphasizing the links between them.

#### *3.2.1 Preimage Study*

This first method shows how to find the directional vector of the minimal parameters studying the structure of the intersection preimage.

Consider a straight line $y = \alpha_0 x + \beta_0$, $0 \leq \alpha_0, \beta_0 \leq 1$. Its digitization according to the Object Boundary Quantization (OBQ for short, see [12] for instance) is the set of discrete points lying on or just under the line: $\{(x, y) \in \mathbb{Z}^2 \mid y = \lfloor \alpha_0 x + \beta_0 \rfloor\} = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq \alpha_0 x + \beta_0 - y < 1\}$.

Now consider a set of pixels $\mathcal{P}$ of a digital line $(a, b, \mu)$ such that $0 \leq a \leq b$. The preimage of $\mathcal{P}$ represents all the Euclidean lines $y = \alpha x + \beta$, $0 \leq \alpha, \beta < 1$ containing $\mathcal{P}$ in their OBQ digitization. This set, denoted $D(\mathcal{P})$, is defined as: $D(\mathcal{P}) = \{(\alpha, \beta) \in [0, 1] \times [0, 1[ \mid \forall (x, y) \in \mathcal{P},\ 0 \leq \alpha x + \beta - y < 1\}$. This preimage lies in a parameter (dual) space $(\alpha, \beta)$ where a point $(\alpha_0, \beta_0)$ maps the line $y = \alpha_0 x + \beta_0$ in the Euclidean space and conversely a line $\beta = x\alpha + y$ in the parameter space maps the point $(x, y)$ in the Euclidean one.

For instance, in this parameter space, the preimage of a infinite digital line of parameters $(a, b, 0)$, with $0 \leq a < b$ is the vertical segment $[(\frac{a}{b}, 0), (\frac{a}{b}, \frac{1}{b})[$: indeed, the set of straight lines containing the digital line in their OBQ digitization is a set of parallel lines of slope $\frac{a}{b}$.

The parameter space and preimage definitions that have been defined for lines with slopes between 0 and 1 can be directly used for lines with slopes between -1 and 0. Conversely, a direct transcription of those definitions for lines with slopes greater than 1 or lower than -1 leads to the definition of another parameter space where a point $(\alpha_0, \beta_0)$ maps to the line $x = \alpha_0 y + \beta_0$ in the Euclidean space.

However, to study the intersection of any two digital lines, we need to work in the same straight line parameter space for any slope. In [13], Veelaert shows that the transformation between those two spaces can be done with a central symmetry in a 3D space. Thus, if the two lines we study have slopes between -1 and 1, we work in the straight line parameter space $P_1$ where a point $(\alpha_0, \beta_0)$ represents the line $y = \alpha_0 x + \beta_0$. If they have slopes greater than 1 or lower than -1, we work in the straight line parameter space $P_2$ where a point $(\alpha_0, \beta_0)$ represents the line $x = \alpha_0 y + \beta_0$. In mixed cases, we use the parameter space $P_1$. In the following the parameter space considered is $P_1$.

In this parameter space, the preimage of a digital straight line of slope $\frac{a}{b}$ with $0 \leq b < a$ and no remainder is the segment $[(\frac{a}{b}, 0), (\frac{a}{b}, -\frac{1}{b})[$. In particular, the preimage of the line of parameters $(1, 1, 0)$ is the vertical segment $[(1, -1), (1, 1)]$ in the parameter space. Some examples are depicted in Figure 4.

Definition 3 can be rewritten using the preimage structure:

**Definition 4** *Let $\mathcal{P}$ be a set of discrete points and $D(\mathcal{P})$ its preimage. The minimal parameters of $\mathcal{P}$ are the values $(\frac{a}{b}, \frac{\mu}{b}) \in D(\mathcal{P})$ such that $b$ and $\mu$ are minimal.*

We consider two digital naive lines $L_1$ and $L_2$ with slopes $\frac{a}{b}$ and $\frac{c}{d}$ and no remainder, and their intersection $I = L_1 \cap L_2$. Without loss of generality, we assume that $\frac{a}{b} < \frac{c}{d}$. We denote $D(L_1)$ (resp. $D(L_2)$) the preimage of $L_1$ (resp. $L_2$). The preimage of any set of discrete points is a convex polygon since it is defined by the intersection of half-spaces. Moreover, the set of pixels $I$ is included in $L_1$ and in $L_2$ which implies that the preimage of $I$ contains $L_1$ and $L_2$ preimages. Those properties imply that $D(I)$ includes the segment $[(\frac{a}{b}, 0), (\frac{c}{d}, 0)]$ (see Figure 5 for illustrations). Furthermore, as $I$ contains all the discrete points belonging simultaneously to $L_1$ and $L_2$, adding one more pixel of $L_1$ or $L_2$ to $I$ cuts $D(I)$ into two parts, one including $D(L_1)$ and the other including $D(L_2)$.

**Theorem 5** *The minimal parameters slope of the intersection of two lines of slopes $\frac{a}{b}$ and $\frac{c}{d}$ ($\frac{a}{b} < \frac{c}{d}$) is given by the rational fraction $\frac{u}{v}$ lying between $\frac{a}{b}$ and $\frac{c}{d}$ with minimal denominator $v$.*

**PROOF.** Consider the set of discrete points belonging to $L_1$ and $L_2$, $I = L_1 \cap L_2$ and call $D(I)$ its preimage. We divide the proof of the theorem into 3 cases that are depicted in Figure 5.

- Assume that $\frac{a}{b} \leq 0$ and $\frac{c}{d} \geq 0$. Then, the fraction $\frac{0}{1}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$. Consequently, the line with slope $\frac{0}{1}$ is a solution, and obviously the solution with minimal denominator (cf. Figure 5(a)).
- Assume that $\frac{a}{b} \leq 1$ and $\frac{c}{d} \geq 1$. Then, the fraction $\frac{1}{1}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$, and from what we said before, we deduce that the line with slope $\frac{1}{1}$ is a solution, and by the way the one with minimal denominator (cf. Figure

5(b)).

- Assume that $0 \leq \frac{a}{b} < \frac{c}{d} \leq 1$. We know that any fraction between $\frac{a}{b}$ and $\frac{c}{d}$ is a solution. By the way, the fraction with minimal denominator lying between $\frac{a}{b}$ and $\frac{c}{d}$ is a solution. We show that there does not exist a solution fraction with a smaller denominator outside the segment defined by $\frac{a}{b}$ and $\frac{c}{d}$. Suppose that there exist such a fraction denoted $\frac{u}{v}$. Then, $v < b$ and $v < d$. Suppose that $\frac{u}{v} < \frac{a}{b}$ and that $|\frac{a}{b} - \frac{u}{v}|$ is minimal for the set of irreducible fractions smaller than $\frac{a}{b}$ with denominator $v$. The case $\frac{u}{v} > \frac{c}{d}$ is symmetrical.

  Consider the discrete point $p(-v, -u-1)$. Adding this point to $L_1 \cap L_2$ implies two new half-spaces constraints given by $0 \leq -\alpha v + u + 1 + \beta < 1$ in the parameter space. This strip is delimited by two lines $l_1 : -\alpha v + u + 1 + \beta = 0$ and $l_2 : -\alpha v + u + 1 + \beta = 1$. $l_1$ cuts the x-coordinate axis for $x = \frac{u+1}{v}$ and $l_2$ for $x = \frac{u}{v}$ (see Figure 5(c)).Thus, since $v$ is smaller than any denominators of the fractions lying between $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{u+1}{v}$ is either greater than $\frac{c}{d}$ or smaller than $\frac{a}{b}$. But since we assume that $\frac{u}{v}$ was the closest fraction with denominator $v$ smaller than $\frac{a}{b}$, we get that $\frac{u}{v} < \frac{a}{b} < \frac{c}{d} < \frac{u+1}{v}$. Finally, $D(I \cup p)$ includes at the same time $D(L_1)$ and $D(L_2)$, which leads to the contradiction.

All the remaining cases can be treated as one of those three. □

### *3.2.2 Geometrical Method*

The preimage study characterizes the value of the minimal directional vector of the intersection of two digital lines. We propose here a geometrical point of view that leads to an algorithm to find both the minimal directional vector and the corresponding remainder.

To do so, we use the structure called Stern-Brocot tree that was defined and studied in section 2.2. Many works deal with the relations between irreducible rational fractions and digital lines. In [14,12], a characterization of the preimage with Farey series is proposed. Indeed, they prove that the preimage of a digital segment has at most 4 vertices whose $x$-coordinates are consecutive terms of a Farey series. In [15], a link between the convex hull of a discrete segment and the decomposition of a fraction into continued fractions is described. In [5], Debled first introduced the link between this tree and digital lines. She noticed that recognizing a piece of digital line is like going down the Stern-Brocot tree until the directional vector of the line is reached. In the following, we call Stern-Brocot tree *root* the two fractions $\frac{0}{1}$ and $\frac{1}{0}$.

**Theorem 6** *Let $L$ be a digital line of slope $\frac{a}{b}$, and $S(\frac{a}{b})$ be the path going from the Stern-Brocot tree root to the fraction $\frac{a}{b}$.*

*Then, for each fraction $\frac{a_i}{b_i}$ lying on $S(\frac{a}{b})$, there exist a subset of $b_i + 1$ pixels* *of $L$ having a minimal directional vector $\frac{a_i}{b_i}$.*

*Moreover, for any other fraction, there does not exist such a subset of $L$.*

This theorem means that the path leading to the fraction $\frac{a}{b}$ represents all the patterns of length smaller than $b$ included in $L$. If $b = 0$ for a given digital line, then we consider the fraction $\frac{b}{a}$ and the same results hold.

The proof of this theorem needs a few lemmas. Lemma 7 was proved by Dorst and Duin in [16].

**Lemma 7** *Let $L_1$ and $L_2$ be two digital naive lines of slope $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ such that $u_2v_1 - u_1v_2 = 1$. Let $C_1$ (resp. $C_2$) be the Chain code associated to a period of $L_1$ (resp. $L_2$) of length $v_1 + 1$ (resp. $v_2 + 1$). Then, the Chain code associated to a period of the digital naive line of slope $\frac{u_1+u_2}{v_1+v_2}$ is $C_1C_2$ of length $v_1 + v_2 + 1$.*

An illustration of this lemma is given in Figure 6(b).

We recall that the *mothers* of a fraction $\frac{u}{v} \in \mathcal{F}_q$ are the two fractions $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ of $\mathcal{F}_{q-1}$ such that $\frac{u_1+u_2}{v_1+v_2} = \frac{u}{v}$. Hence, we have the following result:

**Lemma 8** *Let $\frac{a}{b}$ an irreducible rational fraction and $S(\frac{a}{b})$ its related path. Then, the mothers of $\frac{a}{b}$ lie on $S(\frac{a}{b})$. Moreover, if we denote $A(\frac{a}{b})$ the set of ancestors of $\frac{a}{b}$ according to the definition of mothers, we have $S(\frac{a}{b}) = A(\frac{a}{b})$.*

This lemma is directly derived from the definition and construction of the Stern-Brocot tree.

**PROOF.** [Theorem 6] Let $\frac{a}{b}$ be an irreducible rational fraction and $S(\frac{a}{b})$ its related path. Let $\frac{u}{v} \in S(\frac{a}{b})$ be another rational fraction. Two possibilities:

- if $\frac{u}{v}$ is one of $\frac{a}{b}$ mothers, then we derive the result from lemma 7;
- otherwise, according to lemma 8, $\frac{u}{v}$ is one of $\frac{a}{b}$ ancestors, and the result is obtained by induction.

$\frac{a}{b}$'s ancestors represent all the connected subsets of discrete points that appear in the digital line of slope $\frac{a}{b}$. As $S(\frac{a}{b}) = A(\frac{a}{b})$, there is no fraction corresponding to a connected pattern of the digital line of slope $\frac{a}{b}$ outside the path $S(\frac{a}{b})$. $\square$

Hence, each node of the tree matches with a pattern. Since the intersection of two digital lines is composed of patterns appearing in the two lines, we just have to look for the closest common ancestor of the two corresponding fractions to find the minimal parameters of the intersection.

**Theorem 9** *Let $L_1$ and $L_2$ be two digital lines of slopes $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$. Then, the minimal parameters of $L_1 \cap L_2$ are given by $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ closest common ancestor in the Stern-Brocot tree.*

If the two digital lines studied are such that $b_1 = 0$ and $a_2 = 0$, then the corresponding nodes are the root of the Stern-Brocot tree, and the minimal parameters are any of the two fractions of the root.

Originally, the Stern-Brocot tree defines only the positive irreducible rational fractions. In order to study the intersection of any two digital lines, we generalize this tree adding its negative symmetrical as shown on Figure 6(a).

It is easy to see with the preimage study or the geometrical method that the directional vector found for two digital lines with no remainder is also solution for any remainder. Nevertheless, if the cardinality of the intersection is smaller than the length of the common pattern described by the directional vector found, there exist smaller parameters. In that case, the minimal directional vector can be found among the common ancestors of the two fractions in the Stern-Brocot tree, looking for the one with the smallest denominator greater than or equal to the intersection cardinality minus 1.

Theorems 5 and 9 are equivalent since looking for the closest common ancestor of two fractions is the Stern-Brocot tree is like looking for the fraction with minimal denominator lying between those two fractions. Nevertheless, this geometrical point of view is useful to design an efficient algorithm to determine the minimal directional vector (see next section). Moreover, we show that this method enables to find the minimal remainder associated to this minimal directional vector.

Let us define the following labelling $\mathcal{L}$ of the Stern-Brocot tree nodes: $\mathcal{L}(\frac{a}{b}) = b\mu + a\mu'$. This labelling is constructed recursively as follows:

- $\mathcal{L}(\frac{0}{1}) = \mu$ and $\mathcal{L}(\frac{1}{0}) = \mu'$;
- let $\frac{a}{b}$ be a node and $\frac{u_1}{v_1}$ and $\frac{u_2}{v_2}$ its mothers: then $\mathcal{L}(\frac{a}{b}) = \mathcal{L}(\frac{u_1}{v_1}) + \mathcal{L}(\frac{u_2}{v_2})$.

Each node label thus depends on only two variables. Now let us consider the intersection of two digital lines $L_1(a, -b, \mu_1)$ and $L_2(c, -d, \mu_2)$. Mapping the remainder values with the corresponding nodes labels, we get the following system:

$$\begin{cases} b\mu + a\mu' = \mu_1 \\ d\mu + c\mu' = \mu_2 \end{cases}$$

Hence, we can deduce the values of $\mu$ and $\mu'$, and injecting those values in the label of the node corresponding to the intersection parameters, we get the remainder of the intersection. If this remainder is not an integer, we take its lower integer part if it is a positive number, and its upper integer part otherwise. Figure 7 illustrates this with an example.

#### *3.2.3 Minimal parameters search algorithm*

We propose in this section an algorithm to find the minimal parameters of the intersection of any two digital naive lines using the framework presented in the previous section. Theorem 9 shows that searching the minimal parameters of the intersection of two digital lines and searching the nearest common ancestor of two nodes in a binary tree is the same problem.

In the following, we use the binary code defined in section 2.2 to identify the rational fraction position in the Stern-Brocot tree.

The proposed algorithm is composed of three parts, related two three different problems:

(1) from a given rational fraction, find the corresponding binary code in the Stern-Brocot tree;
(2) from a given binary code, find the corresponding rational fraction;
(3) from two binary codes, find the binary code of the nearest common ancestor.

The two first questions have already been answered in section 2.2 with Algorithms 1 and 2. The third problem have a simple solution that can be directly derived from the solution proposed in [17].

Algorithm 3 describes the different steps to compute the nearest common ancestor (NCA for short) of two rational fractions. After the computation of the two binary codes using Algorithm 1, we check if one fraction is an ancestor of the other. In this case, the NCA is the ancestor. Otherwise, we look for the position $k$ of the first difference between the two codes (the function MSB means Most Significant Bit which is the position of the first non-zero bit). Then, we use Algorithm 2 to compute the fraction which has the same code as $C_1$ (or $C_2$) until position $k$. Algorithm 2 returns the two mothers of the fraction we look for, and thus, a simple median calculus leads to the solution.

Algorithm 4 is the general algorithm to compute the minimal parameters of the intersection of two digital lines. If the two fractions $\frac{a}{b}$ and $\frac{c}{d}$ have different signs, then we know from Proposition 2 that the intersection of the two lines is one pixel or empty, and thus, that the minimal slope is $\frac{0}{1}$. Otherwise, if the two fractions are positive (resp. negative), then the binary codes are computed from the fraction $\frac{1}{1}$ (resp. $-\frac{1}{1}$).

**Algorithm 3** Compute the nearest common ancestor of two rational fractions $\frac{a}{b}$ and $\frac{c}{d}$

NCA$(a, b, c, d)$

1: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
2: $C_1$ = Compute_Code$(a, b)$;
3: $C_2$ = Compute_Code$(c, d)$;
4: **if** ($C_1$ = prefix($C_2$)) **then** {resp. ($C_2$ = prefix($C_1$))}
5:     return$(a, b)$; {resp. return$(c, d)$}
6: **else**
7:     k = MSB($C_1$ XOR $C_2$);
8:     res = Compute_Fraction($C_1$,k,0,I); {res is a $2 \times 2$ matrix}
9:     return(res[1,0] + res[1,1], res[0,0] + res[0,1]);
10: **end if**

**Algorithm 4** Compute the minimal parameters of two digital lines of parameters $(a, b, \mu)$ and $(c, d, \mu')$.

Minimal_Parameters$(a, b, \mu, c, d, \mu')$

1: **if** ($a = c$ AND $b = d$) **then**
2:     return $(a, b, \min(\mu, \mu'))$;
3: **end if**
4: **if** $(\mathrm{sgn}(\frac{a}{b}) \neq \mathrm{sgn}(\frac{c}{d}))$ **then**
5:     $(u,v) = (0, 1)$;
6: **else**
7:     $(u,v) = \mathrm{NCA}(a, -b, c, -d)$;
8: **end if**
9: Find $k$ and $k'$ such that: $\begin{cases} bk + ak' = \mu \\ dk + ck' = \mu' \end{cases}$
10: **if** $(vk + uk' > 0)$ **then**
11:     return$(u, -v, \lfloor vk + uk' \rfloor)$;
12: **else**
13:     return$(u, -v, \lceil vk + uk' \rceil)$;
14: **end if**

Let us look at a run of Algorithm 4 for the two digital lines illustrated in Figure 3: $D_1(4, -5, 0)$ and $D_2(5, -8, 2)$. First, the conditions on lines 1 and 4 are not fulfilled, therefore we have to compute $NCA(4, 5, 5, 8)$ using Algorithm 3. We enter the $NCA$ function, and get $C_1 = 0111$ and $C_2 = 0101$. The first difference between $C_1$ and $C_2$ appears in position 2 (denoted by $k$ in Algorithm 3). Thus, the call (line 8 of Algorithm 3) of the *Compute_Fraction* function presented in Algorithm 2 returns the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Hence we get

$(u,v) = NCA(4,5,5,8) = (2,3)$: $\frac{2}{3}$ is the slope of the minimal parameters line. The resolution of the system on line 9 returns $k = \frac{8}{7}$ and $k' = -\frac{35}{28}$. Finally, we get that $vk + uk' = \frac{26}{28}$ and the final result is $(2,-3,0)$ as depicted on Figure 3.

It is quite easy to see that all the operations of those four algorithms can either be executed in constant time or in linear time in the length of the binary codes. Let us consider an irreducible rational fraction $\frac{a}{b} = [q_0, q_1, \ldots, q_n]$. From the result of section 2.2 Equation 1, we derive that the length of its binary code is $L(\frac{a}{b}) = \sum_{0 \le i \le n} q_i$. So considering two rational fractions $\frac{a}{b}$ and $\frac{c}{d}$, if we denote $M = \max(L(\frac{a}{b}), L(\frac{c}{d}))$, Algorithm 4 has a time complexity in $\mathcal{O}(M)$.

### *3.3 Extent of the intersection*

In order to have a complete description of the intersection of two digital lines, a knowledge of the number of intersection points, some bounds on their coordinates and an enumeration algorithm are required.

As said in the introduction, an enumeration algorithm have been proposed in [6]. This algorithm uses the fact that the intersection of two digital lines is equivalent to a quasi-affine application.

From this algorithm, it is obviously possible to derive the number of intersection pixels. Nevertheless, a straightforward counting of the intersection pixels is possible using a geometrical approach. Indeed, from the definition of naive line, we derive that the intersection of two digital lines can be represented as a parallelogram with rational vertices (intersection of two strips). Counting the intersection pixels is then equivalent to counting the number of integral points in a polygon with rational vertices. A method using Ehrhart polynomials and the related program are proposed in [18].

Finally, it may be also interesting to have bounds on the coordinates of the intersection pixels. Using the same polygonal representation of the intersection as before, finding a bound on a coordinate is equivalent to solving an integer linear program which is generally a NP-hard problem. But since our problem lies in dimension 2, an heuristical algorithm may be designed for this particular optimization problem. Nevertheless the enumeration algorithm proposed in [6] enables to get such an information.

## 4 Digital planes intersection

In this part, we extend the properties found on digital lines intersection for digital planes intersection and present some properties peculiar to planes. A digital naive plane of parameters $(a, b, c, \mu)$ is the set of integer points $\{(x, y, z)\}$ fulfilling the condition $0 \leq ax + by + cz + \mu < \max(|a|, |b|, |c|)$. As for naive lines, digital naive planes are the thinnest 18-connected digital planes without 6-connected holes. See Figure 8(a) for an illustration of the intersection of two naive planes.

### *4.1 Periodicity*

**Proposition 10** *Let $P_1(a, b, c, \mu)$ and $P_2(d, e, f, \nu)$ be two digital planes. Let $v = (v_1, v_2, v_3)^T$ be the cross product of $(a, b, c)^T$ and $(d, e, f)^T$. Let $g = gcd(v_1, v_2, v_3)$ and $v' = \frac{1}{g}v$. Then $P_1 \cap P_2$ is periodic of period $v'$.*

**PROOF.** Let $r_1(x, y, z) = ax+by+cz+\mu$ and $r_2(x, y, z) = dx+ey+fz+\nu$ be the remainder functions of the two planes. Let $(x_M, y_M, z_M)$ be the coordinates of a voxel $M \in P_1 \cap P_2$. It is straightforward to prove that $M + tv'$ is not an integer point if $t$ is not integer. First, let us prove that $r_1(M + v') = r_1(M)$:

$$\begin{aligned} r_1(M + v') &= ax_M + by_M + cz_M + \mu \\ &\quad + \tfrac{1}{g}(abf - ace + bdc - abf + ace - bcd) \\ &= r_1(M) \end{aligned}$$

Similarly, $r_2(M + v') = r_2(M)$. This proves that $M + v' \in P_1 \cap P_2$ which means that $P_1 \cap P_2$ is periodic of period $v'$. □

Since the proof of this proposition does not depend on the thickness of the planes, this result holds for the intersection of any digital planes with any thickness.

### *4.2 Minimal parameters*

In this part, we focus on the minimal parameters of the intersection of two digital planes. To work in the same parameter space for any parameters, we use the same trick as the one proposed by Veelaert [13] for lines, presented in section 3.2.1. Hence, we work in the parameter space $(\alpha, \beta, \gamma)$ where a point

$(\alpha_0, \beta_0, \gamma_0)$ stands for the plane $\alpha_0 x + \beta_0 y + z + \gamma_0 = 0$ in the Cartesian space for any value of $\alpha_0$, $\beta_0$ and $\gamma_0$.

Given two digital planes $P_1$ and $P_2$, we look for the plane parameters $(u, v, w, \mu)$ with minimal $w$ and $\mu$ containing all the voxels of $P_1 \cap P_2$ in their OBQ digitization.

In the following, we consider digital naive planes with no remainder. First of all, Proposition 11 gives a description of the intersection preimage.

**Proposition 11** *Let $P_1(a, b, c, 0)$ and $P_2(d, e, f, 0)$ be two digital naive planes. We denote $I = P_1 \cap P_2$. Then the preimage of $I$, denoted $D(I)$, is a polygon included in the plane perpendicular to $\gamma = 0$ and containing the points $(\frac{a}{c}, \frac{b}{c}, 0)$ and $(\frac{d}{f}, \frac{e}{f}, 0)$.*

**PROOF.** Since the two planes have no remainder, the point $(0, 0, 0)$ is a lower leaning point of the two digital planes. As $I$ is periodic of period $v$ (Theorem 10), for every integer t, the point $tv$ belongs to $P_1 \cap P_2$ and is a lower leaning point of the two digital planes. In the dual space, the point $tv$ corresponds to the two constraints $0 \leq \alpha t v_1 + \beta t v_2 + t v_3 + \gamma < 1$. Since $tv$ is a lower leaning point for the two digital planes, the constraint $\alpha t v_1 + \beta t v_2 + t v_3 + \gamma = 0$ goes through the two points $(\frac{a}{c}, \frac{b}{c}, 0)$ and $(\frac{d}{f}, \frac{e}{f}, 0)$. Hence, for all $t$, $D(I)$ is constrained by the plane $\alpha t v_1 + \beta t v_2 + t v_3 + \gamma = 0$, equivalent to $\alpha v_1 + \beta v_2 + v_3 + \frac{1}{t}\gamma = 0$ for $t \neq 0$. When $t$ increases to $+\infty$, the normal vector of this plane tends to the value $(v_1, v_2, 0)$ with positive values of $t$ and with negative values of $t$ when $t$ goes to $-\infty$. Then, for infinite planes, $D(I)$ is reduced to a polygon included in the plane with normal vector $(v_1, v_2, 0)$ which contains the two points $(\frac{a}{c}, \frac{b}{c}, 0)$ and $(\frac{d}{f}, \frac{e}{f}, 0)$. $\square$

An example of an intersection preimage is given Figure 8(a).

This description enables to characterize the minimal parameters of $I$:

**Theorem 12** *Let $P_1(a, b, c, 0)$ and $P_2(d, e, f, 0)$ be two digital naive planes. We denote $A(\frac{a}{c}, \frac{b}{c}, 0)$ and $B(\frac{d}{f}, \frac{e}{f}, 0)$ the corresponding points in the parameter space, and $I = P_1 \cap P_2$. Then, the minimal normal vector of $I$ is given by the point $(\frac{u}{w}, \frac{v}{w}, 0)$ on $[AB]$ with minimal $w$.*

**PROOF.**

Without loss of generality, we suppose that $\frac{a}{c} \leq \frac{d}{f}$. To prove this theorem, we use the results obtained for digital lines using a digital plane decomposition into digital lines presented in [19]. Indeed, we can decompose any digital plane

$P(a,b,c,\mu)$ into digital 3D lines: for instance, a decomposition along the $y$ axis gives the set of lines $S_{yj}(P) = \{(x_0,y_0,z_0) \in P | y_0 = j\}, \forall j \in \mathbb{Z}$. For two out of these three possible decompositions, those lines are naive lines, and for the third one, they are thicker than naive lines.

Since $I$ is a piece of naive plane, we can use this decomposition. Consider the decomposition of $I$ along the $y$ axis. We denote $S_{yj}(I)$ the 3D digital lines of this decomposition. Then we have $D(I) = \bigcap_j D(S_{yj}(I))$. Moreover, $S_{yj}(I) = S_{yj}(P_1 \cap P_2) = S_{yj}(P_1) \cap S_{yj}(P_2)$ as $S_{yj}(I)$ is the set of pixels of $P_1 \cap P_2$ whose the $y$-coordinate is $j$.

Let us consider the set $S_{y0}(I) = S_{y0}(P_1) \cap S_{y0}(P_2)$. Then, we get two cases:

- if $S_{y0}(P_1)$ and $S_{y0}(P_2)$ are naive lines, we denote them $N_{3D,1}(a,c,0)$ and $N_{3D,2}(d,f,0)$. Then, $S_{y0}(I) = N_{3D,1} \cap N_{3D,2}$.
- otherwise, $S_{y0}(P_1)$ or $S_{y0}(P_2)$ is thicker than a naive line but contains the naive line of the previous case. Thus we have $S_{y0}(I) \supset N_{3D,1} \cap N_{3D,2}$.

If we consider the preimages of those sets, we then get the following property: $D(S_{y0}(I)) \subseteq D(N_{3D,1} \cap N_{3D,2})$.

$N_{3D,1} \cap N_{3D,2}$ is a piece of 3D naive line and its preimage is a prism such that the basis in the plane $\beta = 0$ is the preimage of the intersection of the two 2D naive lines $N_{2D,1}(a,c,0)$ and $N_{2D,2}(d,f,0)$ and such that the directional vector is $(1,0,0)^T$.

Let $p(\frac{u}{w}, \frac{v}{w}, 0)$ be a point of $D(I)$ as illustrated on Figure 8. Then $p \in D(S_{y0}(I))$ and thus $p \in D(N_{3D,1} \cap N_{3D,2})$. The projection of $p$ along the prism previously described onto the plane $\beta = 0$ is the point $proj(p)(\frac{u}{w}, 0, 0)$. $proj(p) \in D(N_{2D,1} \cap N_{2D,2})$ and according to the results about the preimage of the intersection of two digital 2D naive lines, if $w < c$ and $w < f$, then $\frac{a}{c} \le \frac{u}{w} \le \frac{d}{f}$. If $\frac{a}{c} = \frac{d}{f}$, then $\frac{b}{c} \neq \frac{e}{f}$ and the same argument can be applied using a decomposition along the $x$ axis. Otherwise, finally, we derive that, if $w < c$ and $w < f$, thus $p$ belongs to $[AB]$ from the structure of $D(I)$ presented in Proposition 11. This shows that the minimal parameters are to be found on $[AB]$. $\square$

## 5 Conclusion

In this paper, we present new results about the intersection of two digital lines or two digital planes. We introduce criteria to analyze its connectivity and we propose a characterization of the minimal parameters of a given intersection in function of the parameters of the two lines/planes.

Although the properties are enounced and proved for digital naive lines and planes, those results are also true or can be easily transcribed for standard lines (thinnest 4-connected lines without 8-connected holes) or planes (thinnest 6-connected planes without 18-connected holes). For instance, the connectivity results for lines intersections can be adapted transforming any diagonal moving into an horizontal and a vertical one. Moreover, all the results about minimal parameters are based on the intersection preimage features, which depend on the lines or planes preimage shape. But the preimage of a standard line or plane is a translated copy of the preimage of the naive line or plane having the same parameters. So the extension to standard objects is easy.

The transcription of the minimal parameters search algorithm can be easily done for more than two lines. Indeed, the same arguments hold to prove that the minimal slope is the nearest common ancestor of the $n$ rational fractions corresponding to $n$ digital lines.

Those properties will be used for instance in the polygonalization process for digital curves and digital surfaces to define edges and vertices. A complementary study to design an efficient algorithm for the computation of the minimal parameters of digital planes intersection is an interesting perspective for this application.

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