Finding a Minimum Medial Axis of a Discrete Shape is NP-hard

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Abstract

The medial axis is a classical representation of digital objects widely used in many applications. However, such a set of balls may not be optimal: subsets of the medial axis may exist without changing the reversivility of the input shape representation. In this article, we first prove that finding a minimum medial axis is an NP-hard problem for the Euclidean distance. Then, we compare two algorithms which compute an approximation of the minimum medial axis, one of them providing bounded approximation results.

Key words: Minimum Medial Axis, NP-completeness, bounded approximation algorithm.

1 Introduction

In binary images, the *Medial Axis* (MA) of a shape $\mathcal{S}$ is a classic tool for shape analysis. It was first proposed by Blum [2] in the continuous plane; then it was defined by Pfaltz and Rosenfeld in [14] to be the set of centers of all maximal disks in $\mathcal{S}$, a disk being maximal in $\mathcal{S}$ if it is not included in any other disk in $\mathcal{S}$. This definition allows the medial axis to be computed in a discrete framework, i.e., if the working space is the rectilinear grid $\mathbb{Z}^n$. The medial axis has the property of being a *reversible* coding: the union of the disks of MA($\mathcal{S}$) is exactly $\mathcal{S}$.

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In order to compute the medial axis of a given discrete shape $\mathcal{S}$, we first proceed by computing the *Distance Transform* (DT) of $\mathcal{S}$. The distance transform is a bitmap image in which each point is labelled with the distance to the closest background point. For either d_4 or d_8 (the discrete counterparts of the l_1 and l_∞ norms), any given chamfer distance or the Euclidean distance d_E, the distance transform can be computed in linear time with respect to the number of grid points [18,4,7,11]. For the simple distances d_4 and d_8, MA is extracted from DT by picking the local maxima in DT [18,4,16].

Polynomial time algorithms exist to extract MA from DT in the case of the chamfer norms or the Euclidean distance [16,17]. A Reduced Medial Axis (RMA) is presented in [8]: it is a reversible subset of the medial axis, that can be computed in linear time. Despite the fact that the medial axis exactly describes the shape $\mathcal{S}$, it may not be a set with minimum cardinality of balls covering $\mathcal{S}$: indeed, a maximal disk of the medial axis covered by a union of maximal disks is not necessary for the reconstruction of $\mathcal{S}$.

In this article, we investigate the minimum medial axis problem that aims at defining a set of maximal balls with minimum cardinality which cover $\mathcal{S}$. This problem has already been addressed with algorithms that experimentally filter the medial axis [5,15,6,13].

In section 2 we first detail some preliminaries and the fundamental definitions used if the remainder of the paper. Section 3 presents the proof that the minimum medial axis problem is NP-hard. Finally, we compare a greedy approximation algorithm with the approximation algorithm proposed in [15] (Section 4). The greedy approximation algorithm is a first bounded heuristic.

2 Preliminaries and Related Results

First of all, we recall definitions related to the discrete medial axis. Given a metric d, a (open) ball B of radius r and center p is the set of grid points q such that $d(p,q) < r$. In the following, we consider the Euclidean metric, while the extension of the results to other metrics (such as Chamfer norms for example) will be discussed in section 5.

Definition 1 (Maximal ball) *A ball B is* maximal *in a discrete shape $\mathcal{S} \subseteq \mathbb{Z}^n$ if $B \subseteq \mathcal{S}$ and if B is not entirely covered by another ball contained in $\mathcal{S}$.*

Based on this definition, the medial axis is given by:

Definition 2 (Medial axis) *The medial axis of a shape $\mathcal{S} \subseteq \mathbb{Z}^n$ is the set of all maximal balls in $\mathcal{S}$.*

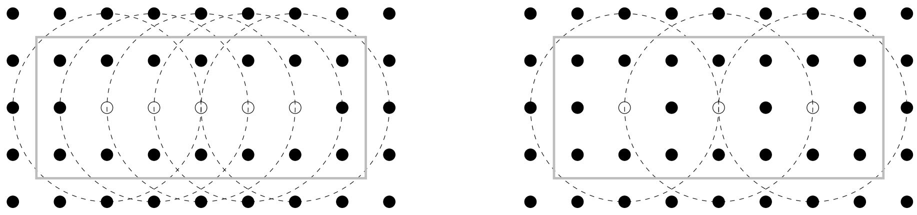

Fig. 1. (*Left*) Unfilled points correspond to the centers of the medial axis balls for the Euclidean metric. In this figure, we represent the discrete maximal balls with the help of their continuous counterpart (open continuous balls) in order to make them distinguishable. (*Right*) A subset of the medial axis the balls of which still cover the entire shape.

In the remainder of the paper, we focus on dimension 2. By definition, the medial axis of a shape $\mathcal{S}$ is a reversible encoding of $\mathcal{S}$. Indeed given the centers and the radii associated to the medial axis balls, the input shape $\mathcal{S}$ can be reconstructed entirely (this process is called the Reverse Distance Transformation [18,3,4,19,8]).

However, this representation is not minimum in the number of balls as illustrated in Figure 1: the set of balls with highlighted centers in the left shape is the medial axis given by Definition 2. However, if we consider the subset of the medial axis depicted in the right figure, we still have a reversible description of the shape with fewer balls. In the following, we define the $k-$medial axis of a shape as follows:

Definition 3 ($k-$Medial axis ($k-$MA)) *A $k-$medial axis of a shape $\mathcal{S} \subseteq \mathbb{Z}^n$ is a subset of the medial axis of $\mathcal{S}$ with k balls which entirely covers $\mathcal{S}$.*

In this paper, we address the problem of finding a subset of the medial axis that still covers all points of $\mathcal{S}$. In the remainder of the paper, we illustrate the proofs with discrete ball coverings of several complex discrete objects. In order to help the reader, we choose to represent each discrete ball with the polygon defined by the convex hull of the grid points inside this ball.

In computational geometry, covering a polygon with a minimum number of a specific shape (*e.g.* convex polygons, squares, rectangles,...) usually leads to NP-complete or NP-hard problems [10]. From the literature, a related result proposed in [1] concerns the minimum decomposition of an orthogonal polygon into squares. At first sight, this result seems to be closely related to the $k-$MAP for the d_8 metric. However, in the discrete case, d_8 balls are centered on grid points and thus have odd widths. Due to this specificity, results of [1] cannot be used neither for the d_8 nor the Euclidean metrics. However, the proof given in the following sections is inspired by this related work.

3 NP-completeness of the k-Medial Axis Problem

Definition 4 ($k-$Medial Axis Problem ($k-$MAP)) *Given a discrete shape $\mathcal{S} \subseteq \mathbb{Z}^2$ of finite cardinality and a positive integer k, does $\mathcal{S}$ admit a $k-MA$?*

In order to prove the NP-hardness of $k-$MAP, we use a polynomial reduction of the Planar-4 3-SAT problem. Let $\phi(V, C)$ be the boolean formula in Conjunctive Normal Form (CNF) consisting of a list C of clauses over a set V of variables. The *formula-graph* $G(\phi(V, C))$ of a CNF formula $\phi(V, C)$ is the bipartite graph in which each vertex is either a variable $v \in V$ or a clause $c \in C$, and there is an edge between a variable $v \in V$ and a clause $c \in C$ if v occurs in c. A *Planar 3-SAT* formula ϕ is a CNF formula for which the formula-graph $G(\phi)$ is planar and each clause is a 3-clause (i.e., a clause having exactly 3 literals).

In the following, we prefer a reduction based on the Planar-4 3-SAT problem: an instance of this problem is an instance of Planar 3-SAT such that the degree of each vertex associated to a variable in the formula-graph is bounded by 4. In other words, a variable may appear at most four times in the boolean formula.

Definition 5 (Planar-4 3-SAT Problem) *Given a Planar-4 3-SAT formula $\phi(V, C)$, does there exist a truth assignement of the variables in V which satisfies all the clauses in C ?*

Planar-4 3-SAT was shown to be NP-complete in [12].

The reduction from any given Planar-4 3-SAT formula ϕ to an instance of $k-$MAP consists in constructing a discrete shape $\mathcal{S}(\phi)$ and finding an integer $k(\phi)$ in polynomial time such that ϕ is satisfiable if and only if $\mathcal{S}(\phi)$ can be covered by $k(\phi)$ balls.

3.1 Variables

Let us first consider a geometrical interpretation of variables. Figure 2 presents a 4-connected discrete object, so called *variable gadget* in the following, defined by the set of grid points below the horizontal dashed line. The eight vertical parts of width 3 of the gadget (numbered on Figure 2) are called the *extremities* of the variable gadget. These extremities are used to plug the "wires" that represent the edges of a formula-graph.

Any minimum covering of this object has 72 balls. This comes first from the fact that all the balls depicted with a thick border belong to any minimum covering; hence 40 balls are required. Moreover, on the remaining part, any two of the 32 circled points (on Figure 2) cannot be covered by a single ball , which proves that at least 72 balls are required to cover a variable gadget. Finally, coverings with exactly 72 balls can be exhibited (see Figure 2), which proves that a minimum covering has 72 balls. Then, if we consider the point p depicted in Figure 2, p can be covered by two different balls, which in turn implies two minimum different coverings. None of these minimum coverings allow protrusions from both one odd extremity and one even extremity. However, one minimum covering allows balls to protrude out at all odd extremities by one row of grid points (Figure 2 top); while another minimum covering allows balls to protrude out at all even extremities also by one row of grid points (Figure 2 bottom). These two coverings mimic the two possible truth assignements of a variable. Without loss of generality, the first covering will correspond to a True assignment, and the other one to a False assignment of the variable.

If the gadget represents the variable x, then each odd extremity carries the literal x, while each even extremity carries the literal $\bar{x}$. A protrusion from a variable extremity can be viewed as a signal 'True' sent from the variable to the clauses. Thus, wires which are used to connect variables and clauses are plugged on odd extremities for positive literals and on even extremities for negative literals.

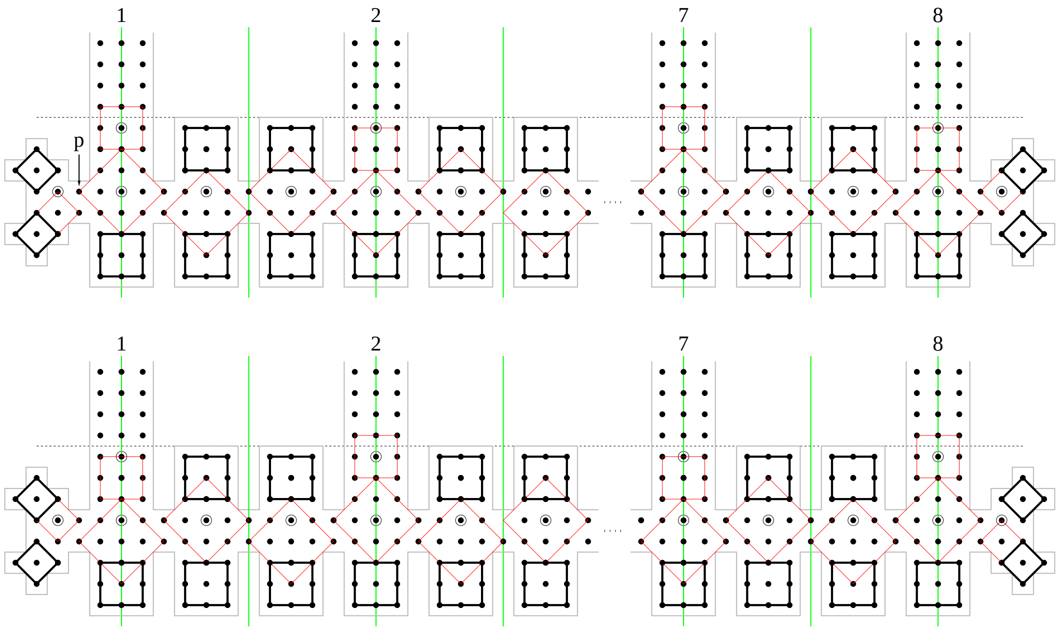

Fig. 2. Two minimum coverings of a variable gadget, corresponding to a True assignement of the variable (top), and False assignement (bottom). Balls with a thick border belong to any minimum covering; any two circled points cannot be covered by a single ball.

Note that this object and its decomposition are invariant under rotation of angle $\frac{\pi}{2}$. Furthermore, the extremities are centered on abscissas with equal values modulo 6 (represented by vertical lines of Figure 2). This property will be used to align the objects and to connect them to each other.

3.2 Wires

In order to connect variables to clauses, we need wires that correspond to edges in the embedding of the formula-graph. A wire must be designed such that it carries either a 'True' signal (protrusion), or a 'False' signal (no protrusion) from variable extremities to clauses without altering the signal (see Fig. 3). We can define a straight wire of width 3 and whose length is equivalent to $0 \mod 3$, so that the signal sent at the left extremity of the wire will be propagated to the right extremity. Furthermore a wire can be bent at angle $\frac{\pi}{2}$ (see Fig. 3). In this case, two minimum decompositions still exist such that if a ball protrudes from one extremity of the wire, then another ball also protrudes out at the other extremity. Furthermore, straight wires and bends can be designed such that the alignment of the abscissa and ordinates of the shape is preserved (*i.e.* is constant modulo 3).

Now, if we consider a complex wire with straight parts and bends, the signals are propagated during the construction of the minimum covering from one extremity to the other one (by induction on the number of bends and straight parts).

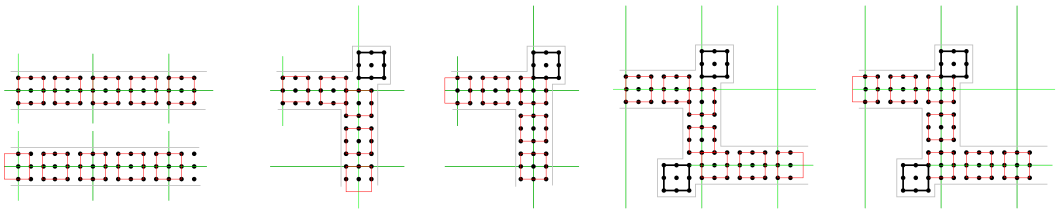

Fig. 3. Wires carrying 'True' or 'False' signals - from left to right: a straight wire, a simple bend, a shift.

3.3 Clauses

Finally, we introduce the *clause gadget*, a component that geometrically simulates a clause. This gadget is the set of grid points to the right of the vertical dashed line in Fig.4. Note that this gadget is not symmetrical because we shall not allow an open ball of radius $\sqrt{8}$ to be placed in its center.

Again, the 5 balls depicted with a thick border belong to any minimum covering. Furthermore, any two of the 5 circled points (on Fig.4, left) cannot be covered by a single ball. Thus, independently covering this gadget requires at least 5+5=10 balls. However, if one open ball of radius 2 is protruding from some wire by one column, carrying a 'True' signal (e.g. the upper one in Fig.4, middle), then minimaly covering the remainder of the gadget can be done with only 9 balls. Similarly, if two or three wires are carrying a protrusion, a minimum covering of the remainder of the clause gadget also has cardinality 9. The case of three protrusions appears on the right in Fig.4, showing that even here 9 balls are still necessary to finish covering the gadget (similarly, any two of the 4 circled points cannot by covered by a single ball). Note that in general there may be several possible minimum coverings of the gadget, although only one is drawn here in each case.

According to these observations, it follows that the clause gadget can be minimaly covered by 10 balls if and only if no input protrusion is observed, in other words if and only if the corresponding clause is not satisfied. Otherwise, if at least one literal of the clause is set to 'True' (protrusion), implying that the clause is satisfied, then only 9 balls are necessary to cover the remainder of the gadget.

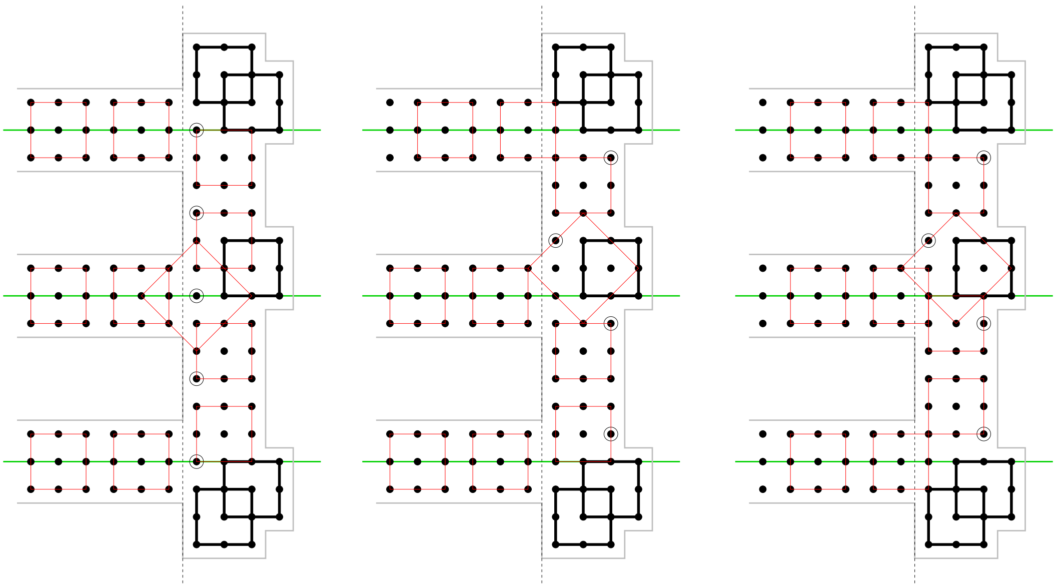

Fig. 4. Three minimum coverings of a clause gadget, depending on the following input signals (from left to right): False-False-False, True-False-False, True-True-True. Balls with a thick border belong to any minimum covering; any two circled points cannot be covered by a single ball.

3.4 Overall Construction and Proof

Given a Planar-4 3-SAT formula $\phi(V, C)$, we are now ready to construct $\mathcal{S}(\phi)$ by drawing a variable gadget for each variable vertex in $G(\phi)$, a clause gadget for each clause vertex in $G(\phi)$, and drawing wires corresponding to the edges in $G(\phi)$, thus linking each literal (the extremity of a variable gadget) to every clause where it occurs.

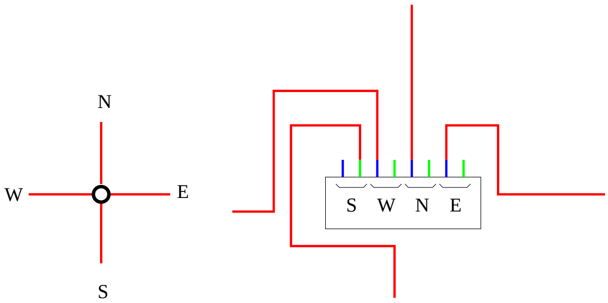

Fig. 5. Illustration of the transformation of a vertex of the planar orthogonal embeding into a variable gadget. In this case, the associated variable appears four times in ϕ, three times as a positive literal, and once as a negative literal.

Lemma 1 *The shape $\mathcal{S}(\phi)$ can be computed in polynomial time in the size of ϕ.*

PROOF. We know from [20] that every planar graph with n vertices (with degree ≤ 4) can be embedded in a rectilinear grid in polynomial time and space. This algorithm produces an orthogonal drawing such that edges are intersection free 4-connected discrete curves. Since our variable gadgets and clause gadgets have a constant size and our wires have constant width, and since ϕ is an instance of Planar-4 3-SAT , it is clear that the construction of $\mathcal{S}(\phi)$ can also be done in polynomial time and space. For example, Figure 5 illustrates how to bend the orthogonal drawing edges in order to connect them to our variable gadget extremities. □

In the following, let $w(\phi)$ denote the minimum number of balls necessary to cover the wires of $\mathcal{S}(\phi)$, and let $k(\phi(V,C)) = 72.|V| + w(\phi) + 9.|C|$.

Lemma 2 *If the formula ϕ is satisfiable, then there exists a covering of $\mathcal{S}(\phi)$ with $k(\phi)$ maximal balls.*

PROOF. Given a truth assignment T of the variables V of ϕ such that ϕ is satisfiable, the following algorithm builds a covering of $\mathcal{S}(\phi)$ with $k(\phi)$ maximal balls:

- cover the variable gadgets according to the truth assignment T ('True' or 'False' value for each variable): each one requires 72 balls allowing protrusions in each extremity carrying a 'True' assignement (Section 3.1);

- cover the wires: since the grid embedding of $G(\phi)$ is computed in polynomial time, so is $w(\phi)$; the protrusions from the extremities of the variables are transmitted to the clause gadgets;
- cover the clause gadgets: since ϕ is satisfiable, at least one input wire of each clause gadget carries a protrusion which implies that 9 maximal balls are enough to cover each clause gadgets (Section 3.3).

Altogether, $72.|V| + w(\phi) + 9.|C| = k(\phi)$ maximal balls are used in this covering. □

Lemma 3 *If there exists a covering of $\mathcal{S}(\phi)$ with $k(\phi)$ maximal balls, then the formula ϕ is satisfiable.*

PROOF. Suppose that there exists a covering of $\mathcal{S}(\phi)$ with $k(\phi)$ maximal balls. By construction, $72.|V|$ plus $w(\phi)$ maximal balls are required to cover the $|V|$ variable gadgets and the wires of $\mathcal{S}(\phi)$. This leaves us with $k(\phi) - 72.|V| - w(\phi) = 9.|C|$ maximal balls to cover the clause gadgets. Since there are $|C|$ clause gadgets, each one is totally covered with 9 maximal balls in the covering, which is possible only if at least one input wire of each clause gadget carries a protrusion (Section 3.3). By construction, this means that the clauses are all satisfied, and in turn that ϕ is satisfiable. □

According to lemmas 2 and 3, there exists a truth assignement of the variables in V which satisfies all the clauses in ϕ if and only if there exists a covering of $\mathcal{S}(\phi)$ with cardinality $k(\phi) = 72.|V| + w(\phi) + 9.|C|$. Thus, if any instance of the $k-$Medial Axis Problem could be solved in polynomial time, then we would have a polynomial time algorithm to solve the Planar-4 3-SAT Problem. Therefore, the $k-$MAP Problem is NP-hard. It is also clear that the $k-$MAP problem is in NP, since we can easily verify in polynomial time wether a set of k balls covers a discrete shape $\mathcal{S}$. Consequently, we have the following theorem:

Theorem 4 *$k-$MAP is an NP-complete problem.*

As a consequence, finding a $k-$MA with minimum k of a shape $\mathcal{S}$ is NP-hard.

4 Approximation Algorithms and Heuristics

Even if the theoretical problem is NP-hard, approximation algorithms can be designed to find the $k-$MA with the smallest possible k. In the literature, several authors have discussed simplification techniques to extract an approximation of the $k-$MA with minimum cardinality [5,15,6,13]. When dealing with

NP-hard problems, we usually want to have bounded heuristics in the sense that the results given by the approximation algorithm will always be at most at a given distance from the optimal solution.

In the following, we first detail the simplification algorithm proposed by Ragnemalm and Borgefors [15] and extended to 3-D by Borgefors and Nyström [6]. Then, we compare their result with a simple but bounded heuristic derived from the MinSetCover problem. These algorithms are presented in a generic way, for any dimension. The experiments are conducted in dimension 3, which is the highest standard dimension for digital objects. Even if the NP-completeness proof is established in dimension 2 in the previous sections, a similar result in dimension 3 can be conjectured.

4.1 Ragnemalm and Borgefors Simplification Algorithm

The algorithm is quite simple but provides interesting results: we first construct a covering map $CM(p) : \mathcal{S} \to \mathbb{Z}$ where we count for each discrete point $p \in \mathcal{S}$, the number of discrete maximal balls containing p. Basically, if a ball B contains a grid point p for which $CM(p) = 1$, then B is necessary to maintain the reconstruction and B belongs to any $k-$MA. Based on this idea, the approximation algorithm can be sketched as follows: let $\mathcal{F} = \mathrm{MA}(\mathcal{S})$, we consider each ball B of $\mathcal{F}$ by increasing radii. If for all points $p \in B$ we have $CM(p) > 1$, then we decide to remove B from $\mathcal{F}$ and we decrease by one the value of $CM(p)$ for each $p \in B$. Then, we process the next ball.

The resulting set $\hat{\mathcal{F}}$ may be such that $|\hat{\mathcal{F}}| < |\mathcal{F}|$. In [15], the author illustrates the reduction rates with several shapes in dimension 2 but no simplification rate is formally given in the general case. In our experiments, instead of considering the medial axis of $\mathcal{S}$, we set $\mathcal{F} = \mathrm{RMA}(\mathcal{S})$ [8].

If $\mathcal{F} = \{B_i, i = 1 \dots k\}$, the overall computational cost of this algorithm is $O(\sum_{i=1}^{k} |B_i| + k \log k)$.

4.2 Greedy Algorithm: a Bounded Heuristic

To have a bounded heuristic, let us consider another problem called the MinSetCover problem [9]: an instance $(\mathcal{S}, \mathcal{F})$ of the MinSetCover consists of a finite set $\mathcal{S}$ and a family $\mathcal{F}$ of subsets of $\mathcal{S}$, such that every element of $\mathcal{S}$ belongs to at least one subset of $\mathcal{F}$. The problem is to find a family of subsets $\mathcal{F}^* \subseteq \mathcal{F}$ with minimum cardinality such that $\mathcal{F}^*$ still covers $\mathcal{S}$. From the optimization MinSetCover problem, we can define the following decision problem: can we cover $\mathcal{S}$ with a family $\mathcal{F}^*$ such that $|\mathcal{F}^*| \leq k$ for a given

k? This decision problem is known to be NP-complete [9]. Replacing $\mathcal{S}$ by a discrete object and $\mathcal{F}$ by the medial axis, we have a specific instance of the MinSetCover problem.

The greedy approximation algorithm is presented in 1. Even if this algorithm is simple, it provides a bounded approximation: if we denote $H(d) = \sum_{i=1}^{d} \frac{1}{i}$, $H_{\mathcal{F}} = H(\max |B|, B \in \mathcal{F})$ and $\mathcal{F}^*$ the $k-$MA, the greedy algorithm produces a set $\hat{\mathcal{F}}$ such that:

$$|\hat{\mathcal{F}}| \leq H_{\mathcal{F}} \cdot |\mathcal{F}^*|$$

Algorithm 1: Greedy algorithm for MinSetCover.

Data: $\mathcal{S}$ and $\mathcal{F}$
Result: the approximated solution $\hat{\mathcal{F}}$
U $= \mathcal{S}$;
$\hat{\mathcal{F}} = \emptyset$;
while $U \neq \emptyset$ **do**
  Select $B \in \mathcal{F}$ that maximizes $|B \cap U|$;
  $U = U - B$;
  $\hat{\mathcal{F}} = \hat{\mathcal{F}} \cup \{B\}$;
return $\hat{\mathcal{F}}$

If we consider $\mathcal{S}$ as a discrete object and $\mathcal{F}$ the medial axis of $\mathcal{S}$, the medial axis simplification problem is a sub-problem of MinSetCover. Hence, Algorithm 1 provides a bounded heuristic for the medial axis reduction and this is, at the time of writing, the only known approximation algorithm for the minimum $k-$MA for which we have an approximation factor. Despite the fact that Algorithm 1 has a computational cost in $O(|\mathcal{S}||\mathcal{F}| \min(|\mathcal{S}|, |\mathcal{F}|))$, a linear in time algorithm can be designed, for instance in $O(\sum_{i=1}^{k} |B_i|)$ [9, Section 37.3].

4.3 Experiments

In Figure 6, we present some experiments of both approximation algorithms. Two observations can be addressed: first, the reduction rate is very interesting since almost half of the medial axis balls can be removed. Secondly, the computational time of both algorithms are similar.

Despite the fact that Ragnemalm and Borgefors's algorithm gives slightly better results, the theoretical bound provided by the greedy algorithm makes this approach a bit more satisfactory.

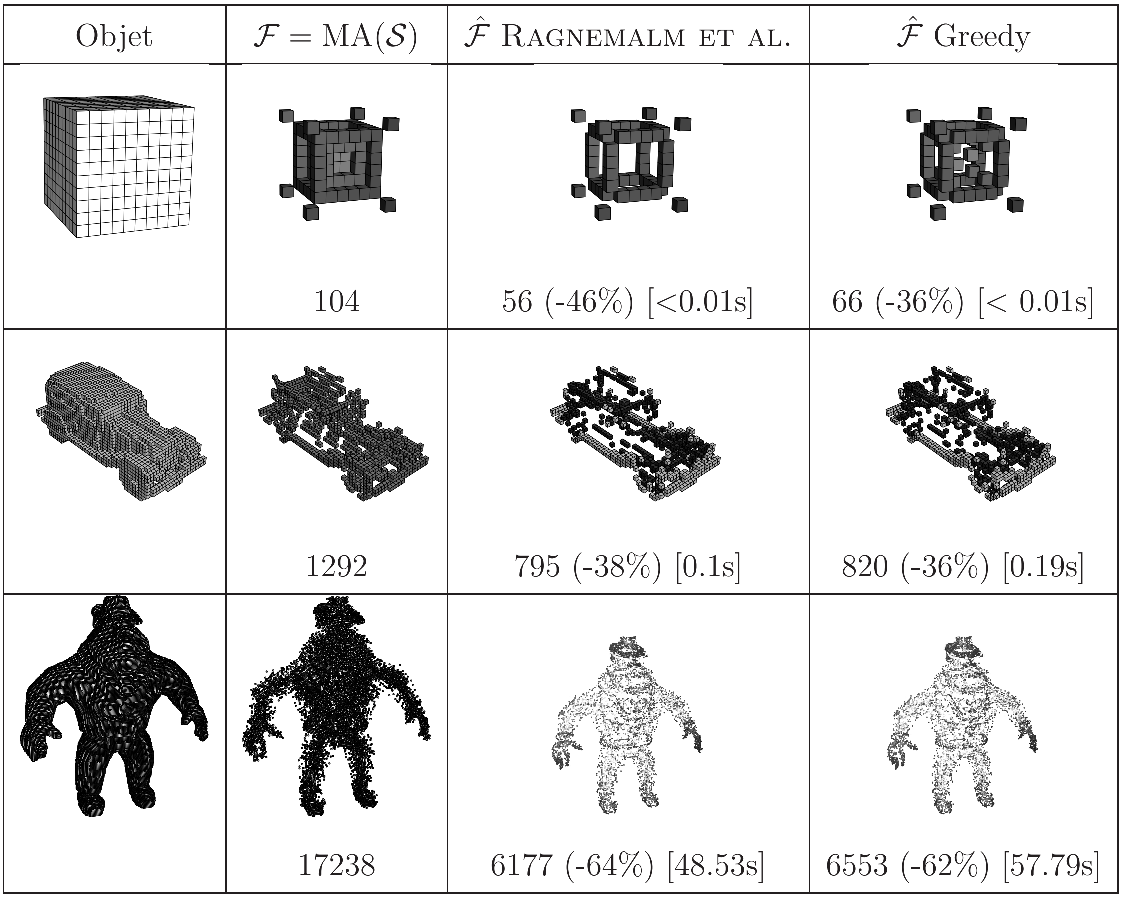

Fig. 6. Experimental analysis of simplification algorithms: *(from left to right)* Discrete 3-D objects, the discrete medial axis (ball centers), simplification obtained by [15] (ball centers), simplification obtained by the proposed greedy algorithm (ball centers). The cardinality of the sets are given below the figure with the reduction ratio (in percent) and the computational time.

5 Discussion and Conclusion

In this paper, we prove that finding a k-medial axis with minimum cardinality k of a discrete shape is an NP-hard problem. To do so, we provide a polynomial reduction from the Planar-4 3-SAT problem to the $k-$MAP. We also experimentally compare the greedy approximation algorithm which provides a bounded approximation, with existing simplification algorithms.

In the proof, we have considered the Euclidean distance based medial axis. In order to derive a proof for the other metrics, new gadgets must be defined. Some cases are trivial, such as the d_8 case for which only the variable gadget must be redefined (see Figure 7). Concerning other metrics, even if the gadgets may be difficult to design, we conjecture that theoretical results may be the same.

Future works concern both the complexity of specific restrictions of the $k-$MAP, and the approximation algorithms. Concerning the theoretical part, the result we give induces the construction of very specific discrete shapes, whose genus depends on the number of cycles in the Planar-4 3-SAT instance. Thus, an important question is whether k-MA is still NP-complete in the case of connected discrete shapes without holes. Concerning approximation algorithms, experiments show that the results of the greedy approximation algorithm are slightly worse than other existing algorithms. An important future work is to merge the two approaches to improve the results while keeping the bounded approximation.

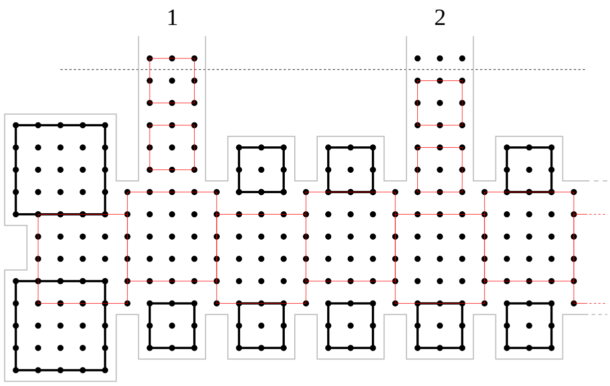

Fig. 7. Outline of a variable gadget for d_8

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